

Infinite-Dimensional Linear Systems Theory

R.F. Curtain

1. INTRODUCTION

Systems theory is the study of mathematical properties of dynamical systems under the influence of external inputs of two distinct types; one which cannot be influenced (the disturbance input) and one which is to be chosen so as to influence the system in some desired manner (the control input). Usually, the performance objective is expressed in terms of the property of an output (the to-be-controlled output) and additional information is available in terms of another output (the observation). In figure 1, we illustrate schematically the general situation of a dynamical system Σ which we aim to influence by another dynamical system Σ_c called the controller. The dynamical system Σ denotes the physical system we wish to influence and the dynamical system Σ_c denotes the controller we seek to design to achieve our objectives. Notice that considered as a dynamical system, Σ_c has as its input the observation, and, as its output, the input to Σ . A typical control problem is to assume that one has a mathematical description of the system Σ , together with how the inputs and outputs interact with it, and to ask how to design a controller Σ_c (in terms of a mathematical description) so as to achieve certain desired performance objectives under the influence of the inputs. It is important to note that everything depends on time, and in mathematical systems theory, both the system Σ and the controller Σ_c are described in terms of time-dependent equations. In most applications, the system is modelled either by a system of coupled ordinary differential

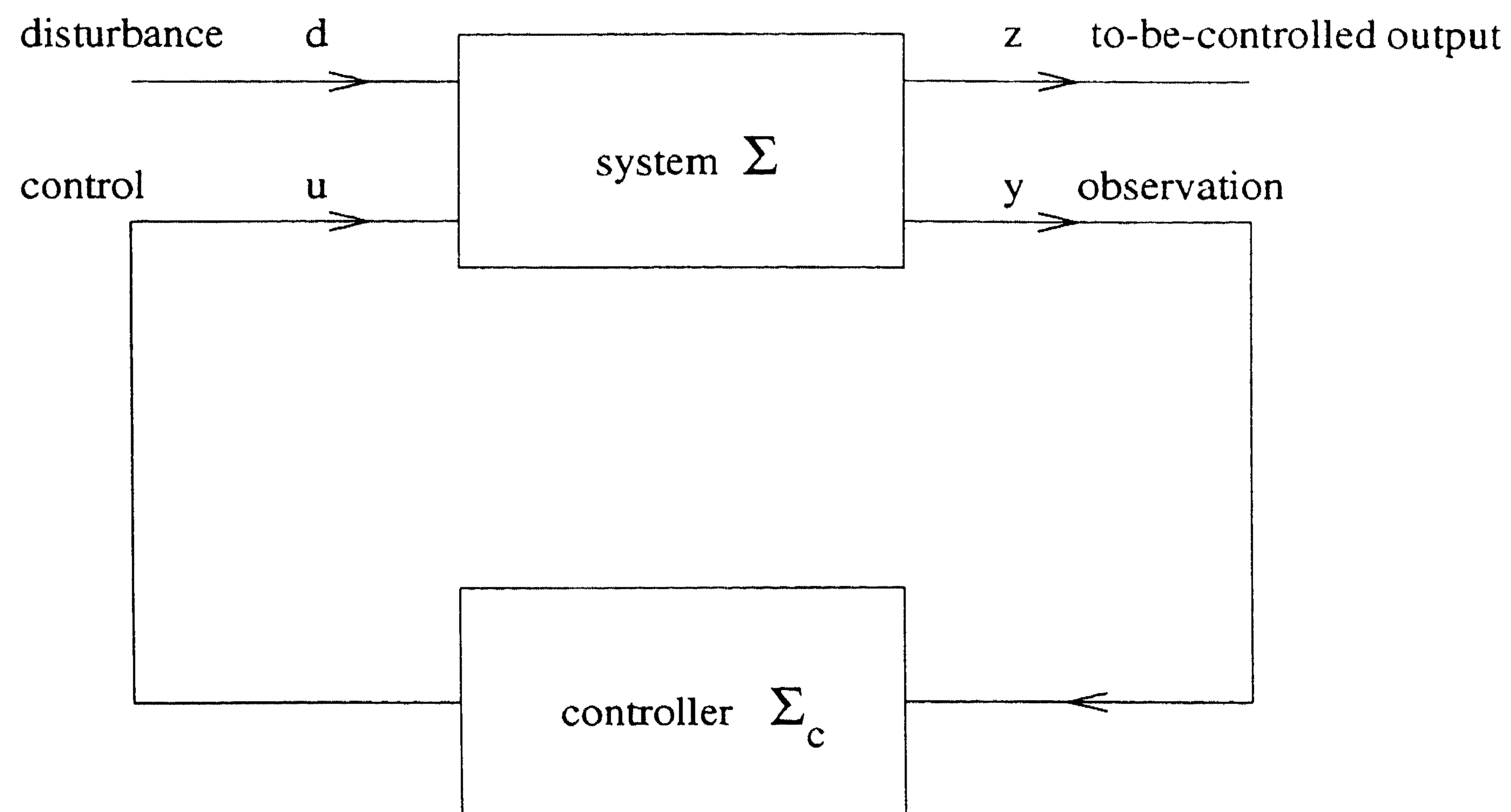


Figure 1. System and controller in closed-loop.

equations (or difference equations) whose solutions depend on the inputs. The outputs are modelled as combinations of the solutions of the ordinary differential equations (or difference equations). Typical problems posed are to design a controller Σ_c as in figure 1, so that the resulting closed-loop system has some desired behaviour; for example, so that the to-be-controlled output z tracks a given trajectory in the presence of unknown disturbances d . The controller produces the input u to the system on the basis of the observation y . An application of such a result would be to devise a strategy to keep a satellite in a prescribed orbit in the presence of atmospheric disturbances.

There are many other problems which have been posed and solved and re-solved in the literature. Moreover, one can pose the same problem for a different class of mathematical descriptions of the system. Not all systems can be adequately described by linear differential or difference equations; many exhibit nonlinear, hysteresis or distributed properties. Below we discuss two particular types of control problems for a particular class of systems, infinite-dimensional linear systems.

2. INFINITE-DIMENSIONAL LINEAR SYSTEMS

Concrete examples are systems described by linear partial differential equations (PDE's) or by delay equations, the solutions of which depend linearly on the inputs. These arise, for example, in modelling the feedback control of large flexible space structures, chemical processes with delayed control action and noise suppression of engines in modern aircraft. The outputs will be linear functionals of the solutions to the partial differential or delay equation. This explains the modifier 'linear' and the 'infinite-dimensional' arises from the fact that the mathematical description is with respect to a Hilbert space, an infinite-dimensional vector space in which the vector represents the state of the system. Often, the term distributed parameter systems is used instead, especially in the engineering literature. In the early sixties there appeared several papers on system theoretic properties, such as controllability and observability, and on some time-optimal control problems. However, two seminal books which had a special impact on the field are [1] by A.G. Butkovskiy and [2] by J.L. Lions. Both considered versions of the classic optimal control problems for systems described by linear partial differential equations, and obtained nice generalizations of the known solutions for ordinary differential equations. In fact, this linear optimal control problem dominated the literature for two decades. One reason for this is that introducing control on the boundary or delayed control action results in a complicated mathematical description that presented an interesting challenge to PDE experts. The motivation of the two research projects described below was to get away from this overstudied problem and to investigate different system theoretic problems which had already had considerable success in the finite-dimensional literature, that is, for systems described by ordinary differential equations. The first project was on *Geometric theory* and resulted in the publication [3]; the second was on *H_∞ -Optimal control theory* and resulted in the publication [4]. It is interesting to compare these topics in one article, because, while both exist as elegant, complete mathematical theories for finite-dimensional linear systems, only one generalizes to a useful theory for infinite-dimensional systems. Fortunately, both are interesting mathematical structures and both have increased our understanding of the now established field of *Infinite-dimensional linear systems theory*.

107

3. GEOMETRIC THEORY

A classic problem in this area is the following disturbance decoupling problem: for the system Σ in figure 1 construct a controller Σ_c such that the input u (depending on the observation y) produces an output z which is independent of the disturbance input d .

If we can achieve such a disturbance decoupling, it clearly has useful applications, for example, in the process industry. Think of Σ as a model of

a distillation column, u as the flow-rate of the liquid stream input, d as the fluctuations in the composition of the feedstream, y as the observed difference in the composition of the products and z as the difference in the composition of the main product. If we could construct a controller Σ_c to produce a time-dependent input u so that z becomes independent of the fluctuations in the composition of the feedstream, this would be an extremely useful device. Clearly, it is too much to expect that one can always achieve disturbance decoupling. So the mathematical problem is to investigate under which conditions this is achievable. For linear finite-dimensional systems there is a very elegant necessary and sufficient algebraic-geometric condition for disturbance decoupling in terms of the system operators (A, B, E, C, D) and a certain (A, B) -invariant subspace, $\mathcal{V}(A, B)$.

Given $n \times n$ and $n \times m$ matrices A and B , we say that a subspace \mathcal{V} of \mathbb{R}^n is (A, B) invariant if $A\mathcal{V}(A, B) \subset \mathcal{V}(A, B) + \text{Im}B$, where $\text{Im}B = \{x \in \mathbb{R}^n : x = Bu, u \in \mathbb{R}^m\}$.

Moreover, (A, B) -invariance can be readily tested and a controller achieving the decoupling can be constructed. It is also possible to design controllers with additional properties, such as the stability of the closed-loop system. An additional pleasing aspect of this geometric theory for finite-dimensional systems is that the mathematical and system theoretic concepts also turn up in completely different control problems, i.e., they have an intrinsic system theoretic significance. While the early work on geometric theory was done in Canada and Italy, some interesting later developments had taken place in The Netherlands, and so it was natural to ask whether any of these problems also have solutions for infinite-dimensional systems. At the time, there was little on this in the literature, but it was clear that all the algebraic properties would carry over. However, there had been some counter-examples in the literature which indicated that there would be problems with the topological aspects. This proved indeed to be the case: the basic catch was that while the (A, B) -invariant subspace \mathcal{V} always exists, it is not always closed, and the disturbance decoupling problem is only solvable if \mathcal{V} is closed (in that case the whole finite-dimensional theory can be generalized). In spite of the lack of a nice generalization, [3] comprises a detailed analysis of the problem, an explanation of the lack of a solution and several examples which give insight into the complex situation for some typical PDE and delay equations. To this day this is the most complete account of geometric control in infinite dimensions. It seems likely that one could obtain a more elegant theory by relaxing the requirement that the output z is completely decoupled from the disturbance to the requirement that it be *almost decoupled*. This, however, remains an interesting conjecture, and a topic for future research.

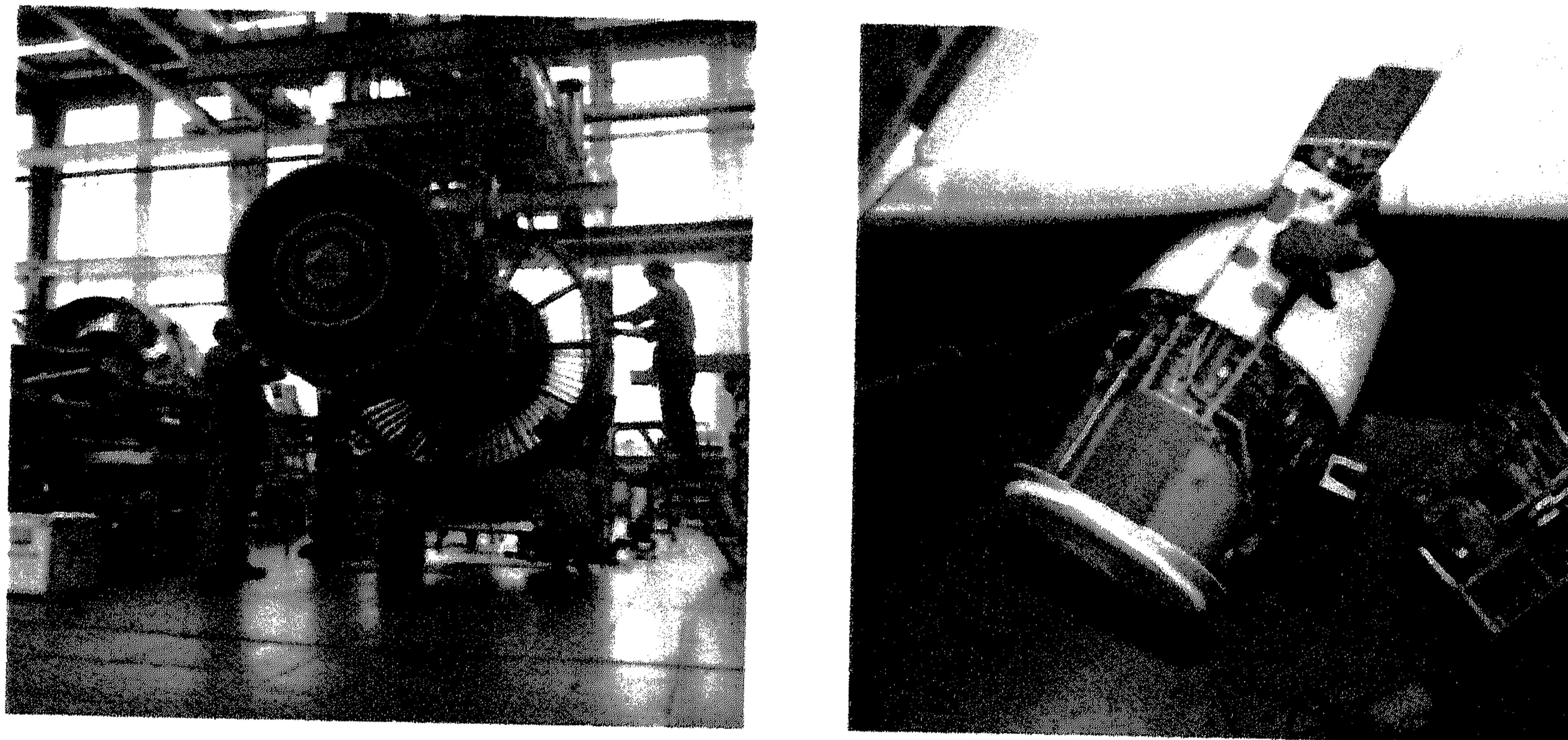


Figure 2. H_∞ -control theory is currently applied to reduce noise in aircraft engines. (Photo Capital Press.)

4. H_∞ -CONTROL THEORY

H_∞ -optimal control problems were formulated for finite-dimensional systems in the 1980's and such problems continue to attract considerable interest to this day. The term H_∞ arose because the mathematical problem involved is the minimization of the norm of a certain operator in the Hardy space H_∞ . A recent application of this theory is to devise an active control mechanism to reduce the noise level created by a new class of powerful engines for aircraft (see [7] and also figure 2). An idealized version of this is the so-called *disturbance attenuation problem*: to design a feedback controller Σ_c for a system Σ as in figure 1 so that the closed-loop system is stable and the influence of the disturbance d on the output z is minimized. This can be seen as a weakened version of the disturbance decoupling problem. Indeed, this problem has a solution under fairly mild conditions and the contribution of the research in [4] was to extend the existing theory for finite-dimensional systems (established in the United States and the United Kingdom) to a class of infinite-dimensional linear systems. Unlike the disturbance decoupling problem, the disturbance attenuation problem has a complete generalization to infinite dimensions. Since the solution involves algebraic operator Riccati equations similar to those occurring in the quadratic control problem in [1], this was not altogether surprising. What was surprising, was the time it took before the infinite-dimensional case was solved. The class of H_∞ -control problems covered by this theory is in fact wider than the disturbance attenuation problem mentioned above.

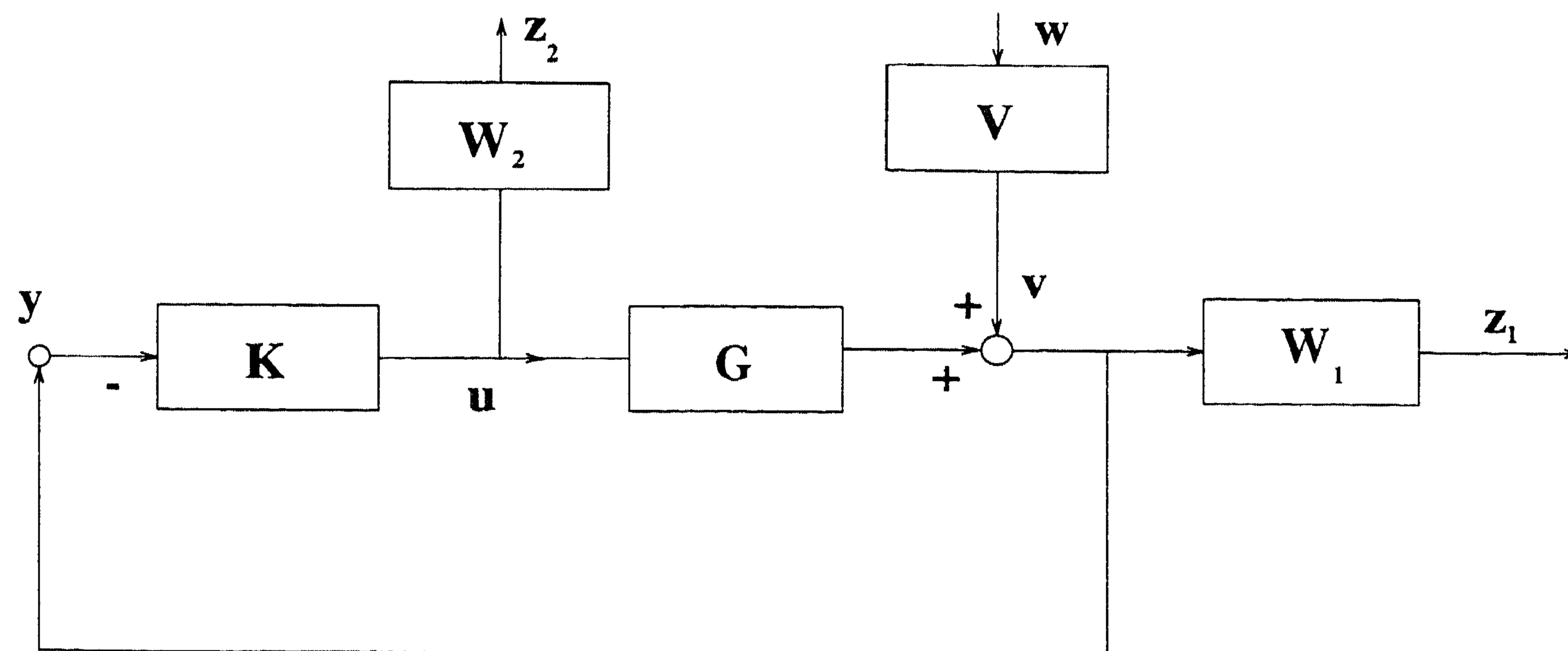


Figure 3. Weighted-mixed-sensitivity design.

Using clever algebraic manipulations, other problems can be formulated just as in figure 1, except that the system Σ is now an abstract system containing the physical system and various weighting transfer matrices, depending on the particular control problem considered. Examples of problems which can be formulated in this abstract way include designing robust controllers, (i.e., controllers which stabilize a whole family of systems) and controllers which also allow for performance objectives such as reduced sensitivity to disturbances in a given frequency band. One such popular controller design is called the *weighted-mixed-sensitivity* design (see [6]). The aim is to design a controller such that in the configuration of figure 3 influence of the disturbance signal w on the output

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

110

is minimized. The weighting transfer matrices W_1, W_2 and V are designed to enhance the performance with respect to robustness and other characteristics of the output. This can be reformulated as a standard problem as in figure 1 with as new extended system Σ_e with transfer matrix

$$\left(\begin{array}{c|c} W_1 V & W_1 G \\ \hline 0 & W_2 \\ \hline -V & G \end{array} \right)$$

G is the transfer matrix of the original system Σ . This is the compelling feature of the H_∞ -formulation; it covers many problems simultaneously. After the publication of [4], or rather at the time of preprints of earlier results, the H_∞ -control problem attracted considerable interest in the United

States, France and Rumania, the aim being to generalize the theory to an even wider class of systems. This achieved some success and research in this area is continuing to flourish. The downside is that the ‘solutions’ are theoretical in nature; the controllers are infinite-dimensional and not readily implementable. More desirable would be algorithms for designing rational controllers for infinite-dimensional systems which have desirable robustness and other properties. To do this one needs approximation results and although some progress in this direction has been made in [5] and [6], much more research remains to be done. Finally, we remark that the solution in [4] is a state-space approach in which the solution is given in terms of operators A, B, C, D . There exist alternative frequency-domain approaches in terms of a transfer matrix description as well. The appeal of the state-space approach is that it reveals the connection with the classical quadratic optimal control problems in [2].

REFERENCES

1. A.G. BUTKOVSKIY (1969). *Theory of Optimal Control of Distributed Parameter Systems*, American Elsevier. (Translation of the original Russian version published in 1965.)
2. J.L. LIONS (1971). *Optimal Control of Systems Described by Partial Differential Equations*, Springer Verlag, Berlin. (Translation of the original French version published in 1968.)
3. H.J. ZWART (1989). *Geometric Theory for Infinite Dimensional Systems*, Lecture Notes in Control and Information Sciences, Vol 115, Springer Verlag, Berlin.
4. B.A.M. VAN KEULEN (1993). *H_∞ -Control for Distributed Parameter Systems: A State-Space Approach*, Birkhäuser, Boston.
5. J. BONTSEMA (1989). *Dynamic Stabilization of Large Flexible Space Structures*, Ph.D. Thesis, Groningen University, The Netherlands.
6. R.F. CURTAIN, Y. ZHOU (1994). *A Weighted-Mixed-Sensitivity H_∞ -Control Design for Irrational Transfer Matrices*, University of Groningen, The Netherlands, Report W-9416.
7. H.T. BANKS, M.A. DEMETRIOU, R.C. SMITH. Robustness studies for H_∞ -feedback control in a structural acoustic model with periodic excitation. To appear in the *Journal of Robust and Nonlinear Control*, R.F. CURTAIN, A. STOORVOGEL (eds.), A Special Issue on Infinite-Dimensional Systems.